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Adaptive Mesh Grading for Finite Element Solutions of an Integral Equation in Quantum Scattering

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This paper investigates adaptive mesh grading in finite element solutions of the two-body Lippmann–Schwinger integral equation. Approximate solutions are obtained from both Galerkin and collocation projection methods using cubic spline approximants. The nodal points are chosen to be equidistributed with respect to a measure that combines both arc length and total curvature.

1. INTRODUCTION

Finite element methods have wide applicability for solving many types of equations. An intrinsic property of the finite element method is the "freedom" in the choice of mesh (nodal points in the case of one dimension and grid points in the case of multidimensions). The correct choice of mesh can improve the accuracy of the numerical solution. In fact, for many problems this choice of mesh may be more important than the particular approximation method.

The basic idea behind adaptive schemes [1] is a procedure for the orderly distribution of points that will optimize the solution with respect to some measure. Often, this measure is chosen to be the arc length or total curvature of the solution function [2, 3]. A weighted combination of arc length and total curvature is the method adopted in this paper. Other examples are truncation errors, or a graded mesh such that the change in the solution function is equidistributed [4]. The procedure may of course be used iteratively. This does not necessarily increase the computer time because the truncation error may be improved so that accurate solutions are obtained with a small number of points. In some sense adaptive mesh

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schemes may be viewed as a problem in control theory [5]: the input is some initial mesh, and with the solution on this mesh, the output is a new mesh. The new mesh in the parlance of control theory is our control function which in the present context yields a good choice of nodal points.

Adaptive mesh schemes are a common tool for solving differential equations [1]. The application of these techniques to the numerical treatment of integral equations is less well understood.

If the mesh is repeatedly subdivided the scheme is usually referred to as an adaptive mesh refinement (AMR). Our approach is to redistribute nodal points and this we shall refer to as an adaptive mesh grading (AMG). The purpose of this paper is to perform a numerical experiment using AMG procedures for solving an integral equation of the second kind

$$f(s) = y(s) - \int_{-1}^{1} K(s, t)\, f(t)\, dt, \qquad -1 \leqslant s \leqslant 1, \tag{1.1}$$

where the inhomogeneous term y and the kernel K are given. Equations of this type arise in potential theory. Our particular example is the two-body Lippmann–Schwinger equation for quantum scattering by a Reid [6] potential. The kernel of the equation contains a Cauchy principal value singularity. Although the integral is defined over a semi-infinite domain, a standard technique of mapping the integration variable may be used to arrive at Eq. (1.1).

For a sufficiently rapidly decreasing potential function (such as the Reid potential) Eq. (1.1) is of the Fredholm type. Several approximation methods [7] are available for solving Fredholm integral equations of the second kind. Our approach is to approximate the solution function f on a subspace of cubic splines [8]. The expansion coefficients are then found by employing either a Galerkin method or a collocation method. As far as we are aware this paper is the first application of AMG to the numerical treatment of integral equations.

Section 2 describes the AMG technique. Our physical problem and the integral equation are described in Section 3. Our numerical results are given in Section 4, and our conclusions in Section 5.

2. Adaptive Mesh Grading

Let $u(s) \in C^2[-1, 1]$ be a cubic B-spline approximation, defined on a partition $\pi_n = \{x_{n,1}, x_{n,2}, ..., x_{n,n}\}$, where $x_{n,i} > x_{n,i-1}$. To be specific, let these ordered points be the nodal points described in the following section. Further, let

$$\bar{a}_i = \int_{x_{n,i-1}}^{x_{n,i}} \left[1 + \left(\frac{du}{ds}\right)^2\right]^{1/2} ds, \qquad i = 2, ..., n, \tag{2.1}$$

$$\bar{c}_i = \int_{x_{n,i-1}}^{x_{n,i}} \left[1 + \left(\frac{du}{ds}\right)^2\right]^{-3/2} \left|\frac{d^2u}{ds^2}\right| ds, \qquad i = 2, ..., n, \tag{2.2}$$

be the arc length and total curvature of u over the subinterval $I_i = [x_{n,i-1}, x_{n,i}]$. Our approach to AMG is to attempt to equidistribute a weighted combination of the arc lengths

$$\bar{a} = \sum_{i=2}^{n} \bar{a}_i \tag{2.3}$$

and total curvatures

$$\bar{c} = \sum_{i=2}^{n} \bar{c}_i. \tag{2.4}$$

Let $h_i = x_{n,i} - x_{n,i-1}$, and let $u(s)$ represent an approximate solution of Eq. (1.1) defined on the mesh π_n. On each interval I_i we compute Simpson's rule approximations

$$a_i = \frac{h_i}{6}[g'_{i-1} + 4g'_{i-1/2} + g'_i] \tag{2.5}$$

and

$$c_i = \frac{h_i}{6}\left[\frac{u''_{i-1}}{(g'_{i-1})^3} + 4\frac{u''_{i-1/2}}{(g'_{i-1/2})^3} + \frac{u''_i}{(g'_i)^3}\right] \tag{2.6}$$

to $\bar{a}_i$ and $\bar{c}_i$, respectively. Here

$$g'_i = [1 + (u'_i)^2]^{1/2}. \tag{2.7}$$

The values of u' and u'' are given by the cubic spline approximation.

The coefficients $\{a_i\}$ and $\{c_i\}$ are subsequently modified by normalization so that

$$\sum_{i=2}^{n} a_i = \sum_{i=2}^{n} c_i = 1. \tag{2.8}$$

Next, we define

$$k_i = (1-\omega)\, a_i + \omega c_i, \tag{2.9}$$

where ω represents some weighting factor. In order to damp extreme values and to increase the interval of influence we define the weighted means quantity

$$\beta_i = \sum_{j=0}^{2l} \rho_j k_{i+j-l}, \qquad i = l+2, \ldots, n-l, \tag{2.10}$$

for inner subintervals and similar but one-sided weighted means for boundary subintervals. The value of l is chosen to fix the width of the interval of influence, while ρ_j is again a weighting factor.

The measure of the arc length and total curvatures is obtained ultimately by integrating β to obtain

$$S_i = \sum_{j=2}^{i} \beta_j, \tag{2.11}$$

with $S_1 = 0$. The table of values obtained from $S_i = S(x_{n,i})$ is used in inverse form, $x_{n,i} = s(S_i)$, by equidistributing the quantities $\{\beta_i\}$ over $n-1$ intervals. We now let

$$S_i^+ = S_n i/(n-1), \tag{2.12}$$

and compute the new nodal points

$$\hat{x}_{n,i} = s(S_i^+) \tag{2.13}$$

by linear interpolation.

3. Integral Equations

In this section we describe a numerical technique for solving integral Eq. (1.1), and show how this technique may be used to solve the physical problem.

A. *Projection Method*

The numerical approximation is a projection method using cubic splines.

The integral equation (1.1) may be written as

$$(\mathscr{I} + \mathscr{K}) f = y, \tag{3.1}$$

where $\mathscr{I}$ is the identity operator and $\mathscr{K}$ the integral operator

$$\mathscr{K} f = \int_{-1}^{1} K(\cdot, t) f(t)\, dt. \tag{3.2}$$

Let π_n be a partition of the interval $[-1, 1]$ by nodal points $\{x_{n,i}\}_{i=1}^{n}$, where $-1 = x_{n,1} < x_{n,2} < \cdots < x_{n,n} = 1$. On this partition, together with the extended points, $x_{n,-2} \leqslant x_{n,-1} \leqslant x_{n,0} \leqslant x_{n,1}$ and $x_{n,n} \leqslant x_{n,n+1} \leqslant x_{n,n+2} \leqslant x_{n,n+3}$, we construct cubic B-splines $\{B_{n,i}\}_{i=0}^{n+1}$ [9]. Each function $B_{n,i}$ is nonzero on the interval $(x_{n,i-2}, x_{n,i+2})$. As trial functions we choose the linear combination

$$\sum_{i=0}^{n+1} \alpha_{n,i} B_{n,i}, \tag{3.3}$$

where it remains to determine the coefficients $\{\alpha_{n,i}\}_{i=0}^{n+1}$.

Consider the residual function

$$r = \sum_{i=0}^{n+1} \alpha_{n,i} (\mathscr{I} + \mathscr{K})\, B_{n,i} - y. \tag{3.4}$$

Let (Ψ, Φ) denote the usual inner product on $C \in [-1, 1]$,

$$(\Psi, \Phi) = \int_{-1}^{1} \Psi(x)\, \Phi(x)\, dx. \tag{3.5}$$

Let $\{\Phi_{n,i}\}_{i=0}^{n+1}$ be our chosen test functions. The coefficients $\{\alpha_{n,i}\}_{i=0}^{n+1}$ can be obtained by solving a system of linear algebraic equations

$$(r, \Phi_{n,i}) = 0, \qquad i = 0, \ldots, n+1. \tag{3.6}$$

In this paper we consider two examples, namely,

(i) $\Phi_{n,i} = B_{n,i}$, which is the Galerkin method, and

(ii) $\Phi_{n,i}(x) = \delta(x - t_{n,i})$, $\{t_{n,i}\} \in [-1, 1]$, which is the method of collocation.

Notice that in the case (ii) the collocation points $\{t_{n,i}\}_{i=0}^{n+1}$ need not coincide with any of the nodal points $\{x_{n,i}\}_{i=1}^{n}$.

B. *Quantum Scattering*

The nonrelativistic scattering of two particles is described by a partial wave k-matrix, $M(q, \kappa)$, which satisfies the integral equation

$$M(q, \kappa) = v(q, \kappa) - \frac{2}{\pi} P \int_0^\infty v(q, q') \frac{q'^2\, dq'}{q'^2 - \kappa^2} M(q', \kappa). \tag{3.7}$$

Here κ is the incident momentum in the center-of-mass frame, q is a momentum variable, and $v(q, q')$ is the Fourier transformed potential. At scattering threshold $\kappa = 0$. In the case of scattering above threshold, $\kappa > 0$, the integral in Eq. (3.7) is evaluated with respect to a principal value prescription (here denoted by the symbol P). The function $M(\kappa, \kappa)$ is real valued and can be expressed in terms of a phase shift $\delta(\kappa)$ by

$$M(\kappa, \kappa) = -[\kappa \cot \delta(\kappa)]^{-1}. \tag{3.8}$$

We map the variable q onto a finite interval by

$$q(x) = \eta \left(\frac{1 + x}{1 - x} \right), \qquad -1 \leqslant x \leqslant 1, \tag{3.9}$$

where η is a constant scale parameter. (Note that η may be viewed as a means of changing the nodal points on $[0, \infty)$. Since we seek an accurate result using a small number of points it is convenient to choose a value of η that yields the best preliminary solution.) Equation (3.7) becomes

$$M(q(x), \kappa) = v(q(x), \kappa) - \frac{4\eta^3}{\pi} P \int_{-1}^{1} \frac{v(q(x), q'(x'))\, M(q'(x), \kappa)}{[(\eta^2 - \kappa^2)(1 + x'^2) + 2(\eta^2 + \kappa^2)x']} \times \left(\frac{1 + x'}{1 - x'} \right)^2 dx'. \tag{3.10}$$

Equation (3.10) has the form of integral equation (1.1).

A complication arises from the principal value integral in Eq. (3.10), and for this we use the method of subtracting the singularity [10]. For simplicity, let $\eta = \kappa$, and consider the moment integral

$$\mathscr{K} B_{n,i}(x) = \frac{\kappa}{\pi} P \int_{x_{n,k-1}}^{x_{n,k}} v(q(x), q'(x'))\, B_{n,i}(x') \left(\frac{1+x'}{1-x'}\right) \frac{dx'}{x'}, \tag{3.11}$$

where $B_{n,i}$ is nonzero at $0 \in [x_{n,k-1}, x_{n,k}]$. Using the subtraction method, we write

$$\mathscr{K} B_{n,i}(x) = \frac{\kappa}{\pi} \int_{x_{n,k-1}}^{x_{n,k}} \left[v(q(x), q'(x'))\, B_{n,i}(x') \left(\frac{1+x'}{1-x'}\right) - v(q(x), \kappa)\, B_{n,i}(0) \right] \frac{dx'}{x'}$$
$$+ \frac{\kappa}{\pi} v(q(x), \kappa)\, B_{n,k}(0) \ln \left| \frac{x_{n,k}}{x_{n,k-1}} \right|. \tag{3.12}$$

The integral in Eq. (3.12) may be evaluated using ordinary numerical quadrature.

4. Numerical Results

This section describes our numerical results. The potential v is chosen to be a nucleon–nucleon Reid [6] $^1S_0$ soft-core potential. This potential is typical of the kind of phenomenological potentials that arise in nuclear physics. In momentum (Fourier) space this potential has the form

$$v(q, q') = \frac{1}{4\mu_1 qq'} \sum_{i=1}^{3} V_i \ln \left[\frac{(q+q')^2 + \mu_i^2}{(q-q')^2 + \mu_i^2} \right], \tag{4.1}$$

where $\mu_1 = 0.7\ \text{fm}^{-1}$, $\mu_2 = 4\mu_1$, $\mu_3 = 7\mu_1$, $V_1 = -10.463\ \text{MeV} \cdot \text{fm}^{-3}$, $V_2 = -1650.6\ \text{MeV} \cdot \text{fm}^{-3}$, and $V_3 = 6484.2\ \text{MeV} \cdot \text{fm}^{-3}$. We take $\hbar^2/m = 41.47\ \text{MeV} \cdot \text{fm}^2$ ($\hbar$ is Planck's constant divided by 2π and m is the particle mass) and set κ^2 to a scattering energy of 24 MeV. The scale parameter is chosen to be $\eta = 10\ \text{fm}^{-1}$.

A reference solution is obtained by solving integral equation (3.7) using the Nyström method [11]. A description for the Nyström method for solving integral equations of the second kind can be found in Ref. [7]. In [11] the points and weights are given by a Gauss quadrature rule that is exact for cubic splines. Figure 1 shows the reference solution $M_{\text{ref.}}(q(x), \kappa)$, where $x \in [-1, 1]$. The phase shift (defined by Eq. (3.8)) is $\delta_{\text{ref.}} = 39.224°$. We denote the error in the approximate phase shift, δ_n, by

$$\Delta\delta = \delta_n - \delta_{\text{ref.}}. \tag{4.2}$$

The phase shift represents a single functional of the solution function. In order to obtain a more global estimate of the error in the approximate solution $M_n(q(x), \kappa)$ we calculate the L^2-norm of the error function

$$\Delta M = M_n - M_{\text{ref.}}, \tag{4.3}$$

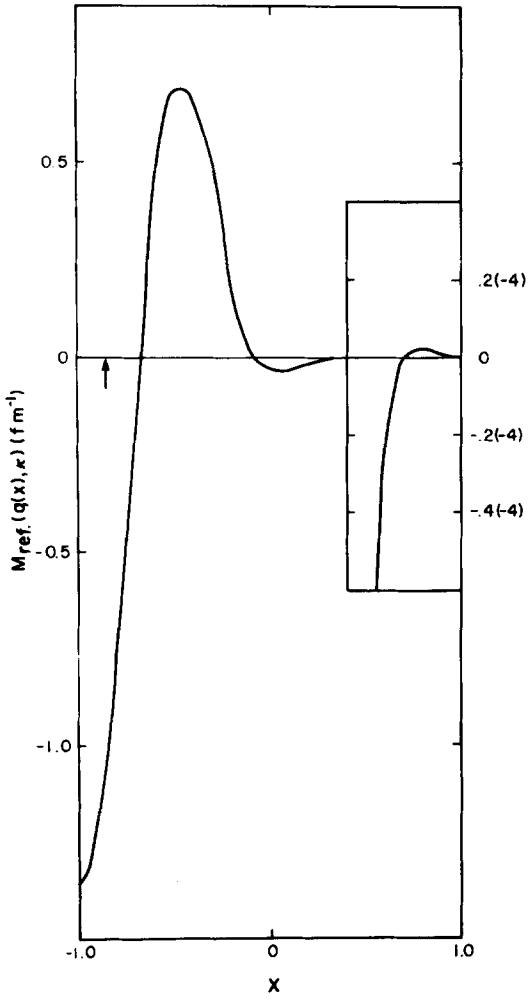

FIG. 1. Reference solution. $q(x) = 10(1+x)/(1-x)$ and $\kappa = 0.7607\,\text{fm}^{-1}$. An arrow marks the on-shell point $q = \kappa$.

where the L^2-norm is defined by

$$\|\Delta M\|_2 = \left[\int_0^\infty |\Delta M(q,\kappa)|^2\, q^2\, dq\right]^{1/2}. \tag{4.4}$$

We remark that although the off-shell function $M(q(x),\kappa)$ is not a physically measurable quantity in two-nucleon scattering, it is required, for example, in few-body [12] calculations. For this reason it is useful to obtain an accurate value of this function.

Our criterion for convergence of the AMG is based on an equidistribution of the quantities $\{\beta_i\}$ (Eq. (2.10)) so that

$$\sigma = \left[\sum_{i=2}^{n} (\beta_i - \bar{\beta})^2 \right]^{1/2} \Big/ (n-1) \tag{4.5}$$

is less than some specified tolerance. Here $\bar{\beta}$ is the average value of $\{\beta_i\}$. On interior subintervals we choose $l=1$, $\rho_0 = \rho_2 = \frac{1}{4}$ and $\rho_1 = \frac{1}{2}$ in (2.10). On boundary intervals we use the relations $\beta_2 = (2k_2 + k_3)/2$ and $\beta_n = (k_{n-1} + 2k_n)/2$.

Our numerical approach is as follows: First, we obtain a preliminary solution of Eq. (3.10) using an initial mesh, which in our case has nodal points given by the formula

$$x_{n,i} = -\cos[\pi(i-1)/(n-1)], \qquad i = 1, ..., n. \tag{4.6}$$

Next we apply AMG iteratively. We denote the number of iterations by I ($I=0$ is the preliminary solution). The nodes that lie outside the interval $[-1, 1]$ are uniformly spaced to the left and right of the interval. These additional nodes which do not affect the spline approximation are left unaltered in our AMG procedure.

We remark that the preliminary mesh (4.6) will reproduce the reference solution provided that a sufficiently fine mesh is used. As an example we quote the error in the phase shift $\Delta\delta$ when $n=18$ which is $-0.004°$ for the Galerkin method and $-0.015°$ for the collocation method.

Our first results concern the Galerkin method. The phase shift error $\Delta\delta$ is tabulated for several choices of n and ω in Tables I–III. Table I shows results using $n=4$ nodes while Table II shows results using $n=10$ nodes. In Table I the AMG gives only marginal improvement in the calculated values of the phase shift and L^2-norm. This in spite of the fact that for $\omega = 0.25$ we obtain a reduction of more than one order of magnitude in σ. In Table II the AMG gives significant improvement (more than one order of magnitude) in the calculated value of the phase shift and L^2-norm.

TABLE I

Galerkin Solution with $n=4$ Using Two Choices of ω

	$\omega = 0$			$\omega = 0.25$		
I	$\Delta\delta$	$\|\Delta M\|_2$	σ	$\Delta\delta$	$\|\Delta M\|_2$	σ
0	−2.0(1)	2.3(−1)	7.3(−2)	−2.0(−1)	2.3(−1)	4.0(−2)
1	−4.9	8.3(−2)	2.0(−2)	−1.3(1)	1.7(−1)	1.2(−2)
2	−2.6	6.3(−2)	1.8(−2)	−1.0(1)	1.4(−1)	4.0(−3)
3	−1.9	6.6(−2)	1.9(−2)	−9.4	1.3(−1)	1.9(−3)
4	−2.7	9.0(−2)	1.9(−2)	−9.0	1.3(−1)	1.1(−3)

Note. Results are shown for the phase shift error $\Delta\delta$ (degrees), the L^2-norm of the error function $\|\Delta M\|_2$, and σ, after I iterates.

TABLE II

Galerkin Solution with $n = 10$

	$\omega = 0$			$\omega = 0.25$		
I	$\Delta\delta$	$\|\Delta M\|_2$	σ	$\Delta\delta$	$\|\Delta M\|_2$	σ
0	−3.5(−1)	1.6(−2)	2.3(−2)	−3.5(−1)	1.6(−2)	2.0(−2)
1	−8.6(−2)	2.3(−3)	5.4(−3)	−7.1(−2)	1.4(−3)	1.1(−2)
2	−2.0(−2)	3.6(−3)	1.9(−3)	−5.1(−2)	1.4(−3)	6.1(−3)
3	−2.6	4.8(−2)	1.4(−3)	−2.9(−2)	1.7(−3)	4.1(−3)
4	1.1(−2)	2.8(−3)	1.7(−3)	−4.4(−3)	1.0(−3)	2.8(−3)
5	3.0(−2)	3.6(−3)	5.3(−4)	−1.5(−2)	1.3(−3)	1.8(−3)

One way to define convergence of the iterative procedure using AMG is to set a tolerance on the value of σ. For example, in Table I a tolerance of $\sigma = 2.0(-2)$ means that the AMG will converge for $I = 1$, that is, after only one application of the AMG. We remark, however, that not all values of σ lead to a converged solution.

The AMG sometimes leads to an oscillatory solution. An example is illustrated in Table III. The result for $n = 4$ using only total curvature ($\omega = 1$) has a solution that oscillates in a 4-cycle with $\Delta\delta = -17$, -16, -18, and -15. The interior nodal points, the L^2-norm and σ are also seen to follow a pattern that repeats after 4 iterations. Clearly, if in this example the tolerance is set to some value $\sigma < 5.6(-2)$ then convergence of the AMG is not possible.

We now consider the behavior of the approximate solution $M_n(q(x), \kappa)$ over the entire domain $[-1, 1]$. Figure 2 illustrates the error ΔM for $n = 4$. We compare the preliminary solution with the $p = 4$ solution using only the arc length ($\omega = 0$). Also

TABLE III

Galerkin Solution with $n = 4$ Using Only the Total Curvature ($\omega = 1$) Showing an Oscillatory Behavior of the Interior Nodes, $\Delta\delta$ (degrees), $\|\Delta M\|_2$, and σ

	Interior nodes				
I	$x_{4,2}$	$x_{4,3}$	$\Delta\delta$	$\|\Delta M\|_2$	σ
20	−3.47(−1)	2.73(−1)	−1.7(1)	2.0(−1)	8.0(−2)
21	−3.33(−2)	5.18(−1)	−1.6(1)	1.9(−1)	5.6(−2)
22	−2.79(−1)	3.23(−1)	−1.8(1)	2.1(−1)	9.2(−2)
23	7.93(−2)	5.77(−1)	−1.5(1)	1.8(−1)	9.0(−2)
24	−3.47(−1)	2.73(−1)	−1.7(1)	2.0(−1)	8.0(−2)
25	−3.33(−2)	5.18(−1)	−1.6(1)	1.9(−1)	5.6(−2)
26	−2.79(−1)	3.23(−1)	−1.8(1)	2.1(−1)	9.2(−2)
27	7.93(−2)	5.77(−1)	−1.5(1)	1.8(−1)	9.0(−2)

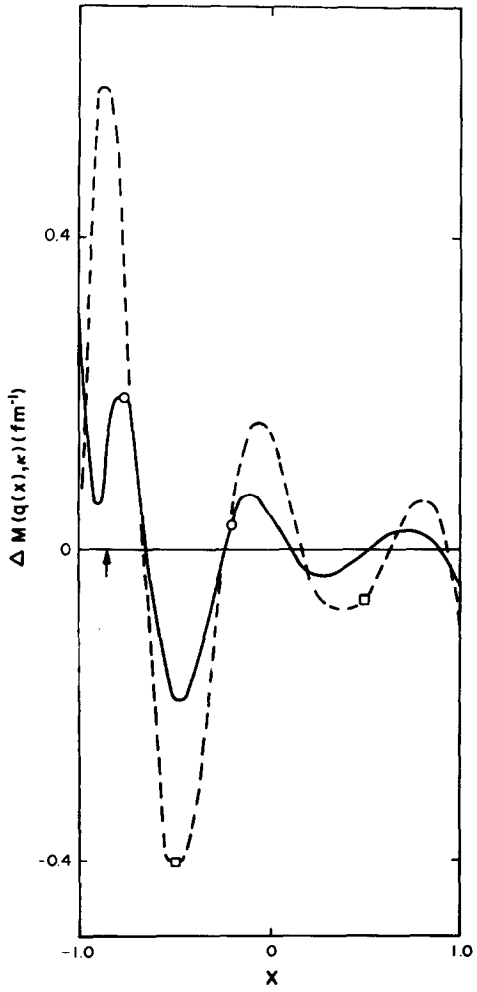

FIG. 2. Galerkin solution for $n = 4$. Broken curve is the preliminary solution. Solid curve is the $I = 4$ solution using arc length. Interior knots are shown for both the preliminary ⊡ and converged ⊙ mesh. An arrow marks the on-shell point $q = \kappa$.

shown are the interior knots: the values of the function at the nodal points. Of course, we do not expect the approximation to reproduce the large momentum tail of the reference solution using such a small number of nodal points. On the other hand, the effect of an AMG using arc length is to move the knots into a region where the solution has large values. Clearly, this result is an improvement on the preliminary solution.

Next we turn to the collocation method. The placement of the collocation points $\{t_{n,i}\}_{i=0}^{n+1}$ is restricted by the Schoenberg–Whitney theorem [8, p. 200,

TABLE IV

Collocation Solution with $n=4$

	$\omega=0$			$\omega=0.25$		
I	$\Delta\delta$	$\|\Delta M\|_2$	σ	$\Delta\delta$	$\|\Delta M\|_2$	σ
0	8.0	2.6(−1)	7.0(−2)	8.0	2.6(−1)	3.9(−2)
1	6.0	1.4(−1)	2.7(−2)	1.4(1)	3.6(−1)	3.5(−2)
2	−3.7	7.7(−2)	1.7(−2)	5.1	1.2(−1)	1.4(−2)
3	−9.3	1.5(−1)	1.5(−2)	3.3(−1)	5.7(−2)	1.2(−2)
4	−1.3(1)	1.9(−1)	1.5(−2)	−3.4	7.2(−2)	8.4(−3)
5	−1.5(1)	2.2(−1)	5.8(−3)	−5.3	9.6(−2)	6.1(−3)

Theorem XIII.1] which requires that $x_{n,i-2} < t_{n,i} < x_{n,i+2}$, $i=0, ..., n+1$. For our preliminary solution we choose the collocation points

$$\begin{aligned} t_{n,0} &= x_{n,1}; \qquad t_{n,1} = \tfrac{1}{2}(x_{n,1} + x_{n,2}); \\ t_{n,i} &= x_{n,i}, \qquad i = 2, ..., n-1; \\ t_{n,n} &= \tfrac{1}{2}(x_{n,n} + x_{n,n-1}); \qquad t_{n,n+1} = x_{n,n}. \end{aligned} \tag{4.7}$$

Consider an AMG for the collocation method. If nodal and collocation points move independently, an interval between nodes may contain no collocation point, a violation of the Schoenberg–Whitney theorem. In this case we can expect a poor result. One way to avoid the problem described above is to fix the collocation points with respect to the nodes by (4.7). This is the approach we have adopted for collocation.

Table V shows results using $n=4$ nodes, while Table V shows results using $n=10$ nodes. Again the result for $n=4$ show little improvement and for $\omega=0$ actually become worse after repeated application of the AMG. Table V shows the same kind of convergence we found using the Galerkin method in Table II.

TABLE V

Collocation Solution with $n=10$

	$\omega=0$			$\omega=0.25$		
I	$\Delta\delta$	$\|\Delta M\|_2$	σ	$\Delta\delta$	$\|\Delta M\|_2$	σ
0	−5.9(−1)	1.8(−2)	2.4(−2)	−5.9(−1)	1.8(−2)	2.3(−2)
1	−1.6(−1)	2.6(−3)	5.9(−3)	−3.6(−1)	4.4(−3)	4.2(−3)
2	−5.3(−2)	3.3(−3)	1.8(−3)	−1.7(−2)	9.9(−4)	1.2(−3)
3	−1.0(−1)	3.7(−3)	8.2(−4)	−1.5(−2)	1.1(−3)	6.3(−4)

5. CONCLUSIONS

We have obtained accurate finite element solutions of the two-body Lippmann–Schwinger integral equation using both Galerkin and collocation methods. Application of the AMG technique can be used to improve the approximate solution by up to one order of magnitude in the computed phase shift and L^2-norm. This represents a significant improvement in the approximate solution of the integral equation.

We remark on some of the difficulties that arise when applying the AMG technique. If a coarse mesh is used then it is often difficult to obtain an improvement in the approximate solution. This is because a small number of nodes may not be able to reproduce the general structure of the solution function. A second difficulty is that for some combinations of arc length and total curvature we find that the nodal points oscillate; in this case the AMG may not reach a converged solution. This result serves as a warning against the blind application of adaptive mesh techniques. However, in those examples where we do find convergence it is noteworthy that the major improvement comes from the first few applications of AMG. This would suggest that AMG is a useful way of improving the finite element solution with little additional cost in computer time.

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